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Jeffery–Hamel boundary-layer flows over curved beds

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We find certain exact solutions of Jeffery–Hamel type for the boundary-layer equations for film flow over certain beds. If β is the angle of the bed with the horizontal and S is the arclength these beds have equation $\sin\beta = (\text{const.})\, S^{-3}$, and allow a description of flows on concave and convex beds. The velocity profiles are markedly different from the semi-Poiseuille flow on a plane bed.

We also find a class of beds in which the Jeffery–Hamel flows appear as a first approximation throughout the flow field, which is infinite in streamwise extent. Since the parameter γ specifying the Jeffery–Hamel flow varies in the streamwise direction this allows a description of flows over curved beds which are slowly varying, as described in the theory, in such a way that the local approximation is that Jeffery–Hamel flow with the local value of γ. This allows the description of flows with separation and reattachment of the main stream in some cases.

1. Introduction

The study of film flow on a solid bed is important in engineering contexts. Lin (1983) lists many references. Fulford (1964) reports applications in chemical engineering and expresses the view that a knowledge of the velocity profiles under various flow conditions would be of great value. Wang (1984) states that 'in many practical cases the bottom plate is not flat but curved, for example the film flow on rollers and wavy plates'. In view of these remarks, and also because of the intrinsic interest of the problem, it seems surprising that little work has appeared on thin-film flows over curved beds. The references known to the present author are all mentioned below.

There is a well-known exact solution of the Navier–Stokes equations for the two-dimensional flow of a film of viscous, incompressible fluid down an inclined plane, with a free surface at constant pressure. This is a steady shear flow with a parabolic velocity profile and is often described as semi-Poiseuille flow. In view of the drastically different types of flow which occur in symmetric curved-wall channels when the Reynolds number multiplied by the small divergence angle is of moderate size, it seems likely that variations to the semi-Poiseuille flow down a plane may occur if the bed is curved. In this paper we obtain some exact solutions for the boundary-layer flow down special curved beds, and later show how those may be applied to a wider class of bed of special type.

Some work has already appeared on flow over curved beds. Wang (1984) has studied thin-film flow down curved beds with Reynolds number of order one. He used an approximate analytic method and included surface-tension effects. The results indicate that flows may occur that are quite different from semi-Poiseuille flow, including a case with separation and reattachment of the main flow. However, he did not give detailed results for velocity profiles, and the range of parameters is restricted. Eagles & Daniels (1986) have studied the boundary-layer approximation to flow down a curved bed, by an elementary approximation similar to Wang's, though they take the series further but neglect some terms included by Wang. They found velocity profiles markedly different from semi-Poiseuille flow for values of the parameters for which the method is useful. However, the limitations on the usefulness of this method are quite severe. Gajjar (1983) has considered the interactive boundary-layer flows down a bed composed of two sections at different (extremely small) angles to the horizontal. The analysis here is very complicated, using multi-scaling methods based on various fractional powers of the Reynolds number. Merkin (1973) and Bertschy, Chin & Abernathy (1983) have performed numerical calculations, based on the boundary-layer equations, for development of the flow from a given velocity profile at some station. Merkin (1973) considered certain special beds whose equations are $\sin\beta = \frac{1}{2}[1-kS/(1+S^2)]$, where S is the arclength and β is the angle with the horizontal. He obtained results for the film thickness and skin friction, but did not give details of the velocity profiles. He stated that there appeared to be a singularity at separation which we shall discuss in §5.

In this paper we consider further the boundary-layer approximation for flow down curved beds. Let the Reynolds number be defined by $R = M/\nu$, where M is the volumetric streamwise flux and ν is the kinematic viscosity. If δ is the dimensionless film-thickness parameter, we assume $R\delta = \lambda = O(1)$ as $\delta \to 0$. The problem then takes the standard form displayed in (2.8)–(2.12). In §3, we point out that for certain special beds of limited streamwise extent, with $\sin\beta = K/S^3$, where K is constant, exact solutions exist. These solutions are just those Jeffery–Hamel solutions appropriate to flow between two planes when the angle between them is small and the Reynolds number is large, except of course that the wedge flow is bisected. These solutions have been extensively described in the context of channel flow by Fraenkel (1962, 1963). Velocity profiles with points of inflexion and with reversed flow near the walls are possible physically realistic solutions amongst the large number of mathematical solutions. The exact solutions of the boundary-layer equations discussed in this paper, like the exact Jeffery–Hamel solutions for flow in a wedge, are not necessarily attainable in practice. However, Fraenkel (1963) has shown that a family of these solutions provides a valid first approximation at every station in certain channels with extremely small wall curvature.

This leads us, in §4, to consider beds of a special type, on which that same family of Jeffery–Hamel solutions appears as the first approximation. In fact we show that at every station the solution of (4.12) for the stream function gives the first approximation. This enables us to describe a variety of interesting flows, including some with separation and reattachment of the main stream within an overall smooth approximation to a solution of the Navier–Stokes equations. There are no singularities in the flow field unless the product of the Reynolds number and the streamwise derivative of the film thickness reaches a certain critical value, which is larger than the value at which the main flow separates from the wall.

2. The governing equations

Consider film flow down a plane bed at angle β to the horizontal (see figure 1). There exists an exact semi-Poiseuille flow solution of the incompressible Navier–Stokes equations, satisfying exactly the conditions on the stress at the free surface,

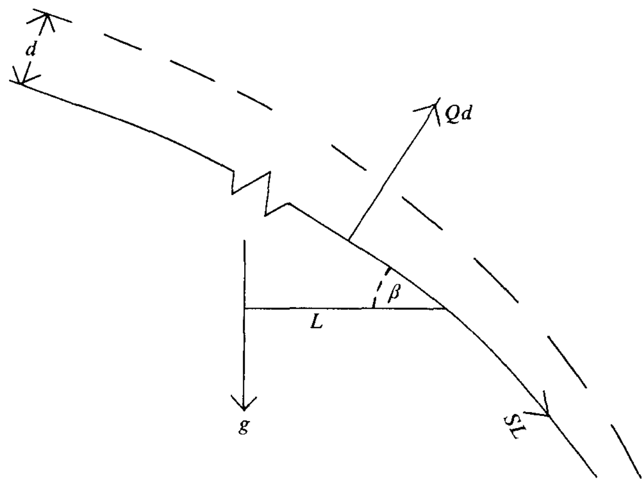

FIGURE 1. Schematic diagram of the geometry. The dashed curve represents the free surface. Lengths are shown in dimensional terms.

where the pressure is assumed to be a constant p_0. The film thickness is d, a constant. Let the volumetric flow rate be M and define a Reynolds number R and a Froude number F by

$$R = \frac{M}{\nu}, \quad F^2 = \frac{M^2}{gd^3}, \tag{2.1}$$

where ν is the kinematic viscosity and g is the gravitational constant. It follows from the exact solution that

$$\frac{R \sin \beta_0}{F^2} = 3. \tag{2.2}$$

We now use a more general bed whose equation is given by

$$\beta = h(S), \tag{2.3}$$

where β is the acute angle made by the bed with the horizontal, and S is the arclength made dimensionless by an arbitrary reference length L. Assume $\beta \rightarrow \beta_0 > 0$ as $S \rightarrow -\infty$, then the exact semi-Poiseuille solution is approached as $S \rightarrow -\infty$ with film 'thickness' d and volumetric flow rate M. Let the dimensionless film-thickness parameter be

$$\delta = \frac{d}{L}, \tag{2.4}$$

and let Q be the perpendicular distance from the bed in units of d. The usual boundary-layer assumptions amount to

$$R\delta = \lambda = O(1) \quad \text{as} \quad \delta \rightarrow 0 \tag{2.5}$$

and

$$\Psi = \Psi(S, Q), \tag{2.6}$$

where Ψ is a stream function non-dimensionalized by M. Standard methods and a careful consideration of the stress conditions at the free surface,

$$Q = G(S), \tag{2.7}$$

lead to the problem

$$\Psi_{QQQ} = \lambda(\Psi_Q \Psi_{QS} - \Psi_S \Psi_{QQQ}) - \frac{3 \sin\beta}{\sin\beta_0}, \tag{2.8}$$

with boundary conditions

$$\Psi = 0, \quad \Psi_Q = 0 \quad \text{on } Q = 0, \tag{2.9}$$

$$\Psi_{QQ} = 0, \quad \Psi = 1 \quad \text{on } Q = G(S), \tag{2.10}$$

$$\Psi \to \tfrac{3}{2}Q^2 - \tfrac{1}{2}Q^3 \quad \text{as } S \to -\infty, \tag{2.11}$$

$$G \to 1 \quad \text{as } S \to -\infty. \tag{2.12}$$

In obtaining this problem we have made use of the second boundary-layer equation, $\partial P/\partial Q = 0$ and boundary condition $P = p_0$ on $Q = G(S)$ to show $P = p_0$ throughout the film. We also used (2.2). The dimensionless fluid velocity components are $u = \partial\Psi/\partial Q$ and $v = -\delta\partial\Psi/\partial S$ within this approximation.

The system (2.8)–(2.12) may be solved numerically by several schemes. See, for example Eagles & Smith (1980) for calculations on a similar problem for channel flow. Merkin (1973) has calculated film thicknesses for certain beds of special form with an initial parabolic profile imposed at $S = 0$ in some cases. Eagles & Daniels (1986) have obtained approximate solutions to the system by expanding Ψ and G in powers of λ, and report that velocity profiles quite substantially different from semi-Poiseuille flow may be attained.

3. Reduction to a Jeffery–Hamel problem for special beds

In this section we show that for a particular class of beds the problem (2.8)–(2.10) admits exact solutions governed by an ordinary differential equation, which is a limiting case of the Jeffery–Hamel equation. We shall then be able to describe in detail a wide variety of possible flows. We introduce the cross-stream variable

$$\eta = \frac{Q}{G(S)} \tag{3.1}$$

and, now regarding Ψ as a function of S and η, we obtain from (2.8)–(2.10)

$$\Psi_{\eta\eta\eta} + \lambda G'(S)\, \Psi_\eta^2 = \lambda G(S)\,(\Psi_\eta \Psi_{\eta S} - \Psi_S \Psi_{\eta\eta}) - \frac{3G^3(S) \sin\beta}{\sin\beta_0}, \tag{3.2}$$

$$\Psi_{\eta\eta} = 0, \quad \Psi = 1 \quad \text{on } \eta = 1, \tag{3.3}$$

$$\Psi = 0, \quad \Psi_\eta = 0 \quad \text{on } \eta = 0. \tag{3.4}$$

The following is a special case that allows exact solutions. We observe that if $G'(S)$ and $G^3(S) \sin\beta$ are both constants, then we may treat Ψ as a function of η only, to obtain an ordinary differential equation. This may be achieved as follows. Let

$$G^3(S) = \frac{\mu^3 \sin\beta_0}{\sin\beta} \tag{3.5}$$

and let

$$G = kS, \quad S_1 < S < S_2, \tag{3.6}$$

where β_0, μ, k, S_1 and S_2 are constants. Then

$$\sin\beta = \frac{\mu^3 \sin\beta_0}{k^3 S^3}, \quad S_1 < S < S_2, \tag{3.7}$$

which specifies a shape of bed within a range of S such that $0 < |\sin\beta| < 1$, provided the constants are suitably chosen.

We now write $\Psi = F(\eta)$ and

$$\lambda k = \gamma \tag{3.8}$$

so that system (3.2)–(3.4) reduces to

$$F'''(\eta) + \gamma\{F'(\eta)\}^2 + 3\mu^3 = 0, \tag{3.9}$$

with boundary conditions

$$F''(1) = 0, \quad F(1) = 1, \quad F(0) = 0, \quad F'(0) = 0. \tag{3.10}$$

If we specify the value of $\gamma = k\lambda$ (which can be done, for example, by specifying R, δ and $\mathrm{d}G/\mathrm{d}S = k$) then we have a third-order differential equation with four boundary conditions and one unknown constant μ. The problem may be solved and the shape of the bed determined from (3.7).

In fact this problem is just that which arises for symmetric flow between inclined planes in the limit with $R\epsilon = \gamma = O(1)$ as $\epsilon \to 0$, where the planes are mutually inclined at an angle 2ϵ, although in that case the constant $3\mu^3$ in (3.9) is related to the streamwise pressure gradient. For a given value of γ many solutions exist. They are limiting cases of the exact solutions for flow between inclined planes. Here we concentrate on the simpler solutions which are connected smoothly with Poiseuille flow in a range of γ including zero. Of course, when $\gamma = 0$ the solution is $F(\eta) = \frac{3}{2}\eta^2 - \frac{1}{2}\eta^3$ with $\mu = 1$.

The problem (3.9) and (3.10) was solved numerically for $-4 < \gamma < 5.45$, and results were checked against earlier calculations using elliptic functions (Eagles 1966). In figure 2 we give a graph of μ^3 versus γ, and in figure 3 we show the velocity profiles $F'(\eta)$ for $\gamma = -4$, 4.71 and 5.3, while in table 1 we give some numerical values of μ^3 for various values of γ.

There are three main classes of flow described by this method, which we list below. (The labelling does not correspond to Fraenkel's numbering, since we are not concerned here with representations in terms of elliptic functions.) We note that in all cases $G \geqslant 1$, $\delta > 0$, $R\delta = \lambda > 0$ and $\sin\beta_0 > 0$.

Class A

(corresponds to Fraenkel's class III_1)

Here we choose $\gamma < 0$. Since $\gamma = \lambda k$ then $k < 0$. Since $\gamma < 0$ then figure 2 shows that $\mu > 0$, and since $G = kS$ then we must take $S < 0$. From (3.7), $\sin\beta > 0$ and $\mathrm{d}\beta/\mathrm{d}S > 0$. The bed is downward sloping throughout and convex upwards. The velocity profile is flatter than semi-Poiseuille flow and the film thickness decreases as S increases. This is like (bisected) flow into a convergent wedge.

An example is with $\gamma = -4$, $\mu = 1.425$, $k = -1.131$, $\lambda = 3.537$, and with $\sin\beta_0 = \frac{1}{64}$ the equation of the bed is $\sin\beta = -\frac{1}{32}S^{-3}$ with, say, $-1 < S < -\frac{1}{4}$. The film thickness in units of L is $1.131\delta|S|$.

Class B

(part of Fraenkel's class I)

We choose $0 < \gamma < 1.81$. Then from figure 2 we see that $0 < \mu < 1$. Since $\gamma = \lambda k$

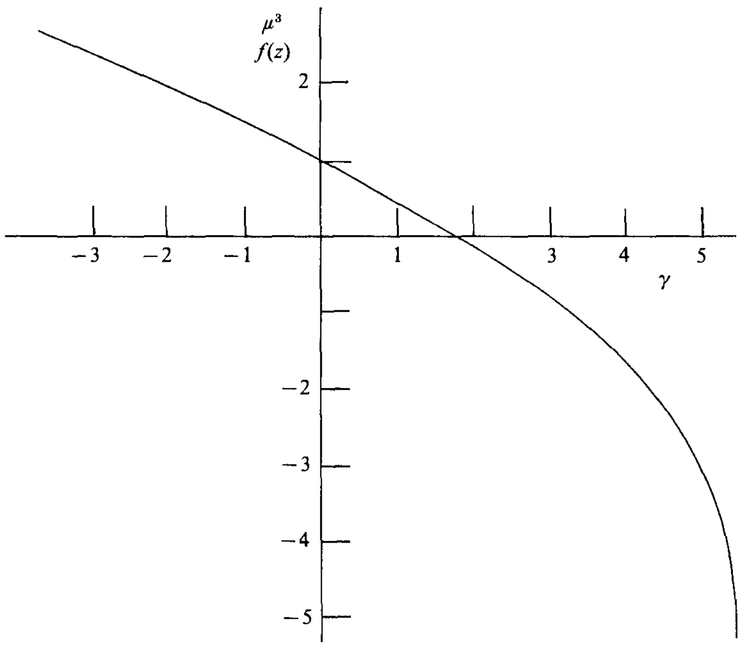

FIGURE 2. Values of μ^3 versus γ obtained by solving (3.9) and (3.10). The ordinate may also be used for finding $f(z)$ in (4.12).

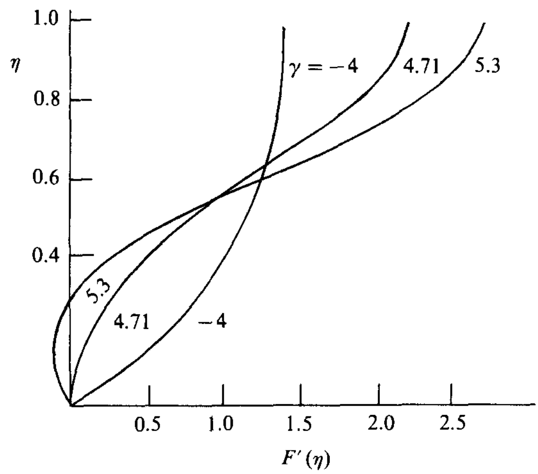

FIGURE 3. Jeffery–Hamel velocity profiles from (3.9) and (3.10), for $\gamma = -4$, 4.71, 5.3.

then $k > 0$, and thus from (3.6) we must take $S > 0$. From (3.7), $\sin\beta$ is positive and $\mathrm{d}\beta/\mathrm{d}S$ is negative, and therefore the bed is downward sloping throughout and concave. The velocity profile is sharper than semi-Poiseuille flow, but has no point of inflexion, and the flow is like the (bisected) flow out of a divergent wedge.

An example is with $\gamma = 1.6$, $\mu = 0.5024$, $k = 0.5024$, $\lambda = 3.184$. The equation of the bed is $\sin\beta = \sin\beta_0/S^3$ with $1 < S < 4$ say, and the film thickness is $0.5024S\delta$.

Class C

[includes part of Fraenkel's class I for $\gamma < 2.988$, class II_1 ($2.988 < \gamma < 4.71$) and class II_2 ($4.71 < \gamma < 5.46$)]

γ	-4	-2	-1	0	0.8	1.6
μ^3	2.893	1.979	1.500	1	0.5775	0.1268
γ	2.4	3.2	4.0	4.8	5.2	5.45
μ^3	-0.3662	-0.9286	-1.623	-2.654	-3.597	-5.314

TABLE 1. Some values of μ^3 obtained by solving (3.9) and (3.10)

We choose $1.81 < \gamma < 5.46$. Then figure 2 shows $-1.74 < \mu < 0$. Since $\gamma = \lambda k$, then $k > 0$, and therefore from (3.6) we must take $S > 0$. From (3.7) we see $\sin\beta < 0$, and $\mathrm{d}\beta/\mathrm{d}S > 0$, so that the bed is upward sloping and convex (it must be envisaged that this is joined to some more general bed so that the slope of the bed is downwards at $S = -\infty$). Each possible velocity profile will have an inflexion point in $0 < \eta < 1$, and for $\gamma > 4.71$ it will have a region of reversed flow near the bed.

An example is $\gamma = 4.8$, $\mu^3 = -2.654$, $k = 1.099$, $\lambda = 4.367$. The equation of the bed is $\sin\beta = -2\sin\beta_0/S^3$ with $1 < S < 4$ say, and $\sin\beta_0 < \frac{1}{2}$.

Whether or not these exact solutions of the boundary-layer equations may be closely attained experimentally is not known. It may be that if a general bed, making an angle β_0 with the horizontal at $S = -\infty$ is joined fairly smoothly to one of the special beds of class A, B or C, the Jeffery–Hamel flows may be attained to a good approximation. Indeed, in the case of a particular slender curved-wall channel joined smoothly to a section of straight-walled channel, where the Jeffery–Hamel solutions are applicable, it is known that these Jeffery–Hamel solutions are approached reasonably quickly as the calculation proceeds downstream (Eagles & Smith 1980). Since the calculation in the present case may be made in the same way (see §5) the same situation would occur here. We are not aware of any experimental results on this matter.

However, bearing in mind that Fraenkel (1963) produced a theory for channels of infinite length and very small wall curvature in which the appropriate Jeffery–Hamel profiles were the correct first approximation at every station, it seems desirable to find such a theory for the thin-film flows. This is achieved in the next section. Although the theory is similar to Fraenkel's in some ways, the details are different, and the method here introduces the new idea of a bed of special type which allows the Jeffery–Hamel solutions to appear within a consistent scheme.

4. Jeffery–Hamel flow over some more general beds

In this section we shall find some beds with the property that the Jeffery–Hamel solutions of §3 provide a first approximation to the flow everywhere, the parameter $\gamma(z)$ of (4.12) varying smoothly as S varies from $-\infty$ to ∞. We shall introduce the ideas by using the boundary-layer equation, but afterwards will note that the approximations so obtained are in fact correct approximations to the full Navier–Stokes equations if the film thickness is small enough.

It is convenient to change the definitions of the variables and parameters slightly in this section. We recall that in §2, d was defined as the film thickness at $S = -\infty$, and $F^2 = M/gd^3$ was based on this. However, the theory to be presented here appears in the most natural and easily grasped form if we proceed as follows. Let

$$d_1 = \text{minimum thickness of film.} \tag{4.1}$$

This may be at $S = \pm\infty$ or at some finite value or values of S. The definition in (2.1) of R is unchanged, and we may define a new Froude number by

$$F_1^2 = \frac{M^2}{d_1^3 g}. \tag{4.2}$$

Of course, if the minimum thickness is at $S = -\infty$ then $d = d_1$ and $F = F_1$. All the definitions of §2 are now to be the same except that d is replaced by d_1. Strictly, of course, we should rename the parameters and variables, but to avoid this notatonal complication we keep the same symbols for the variables Q and η and for the parameters δ and λ.

We introduce the notation

$$\frac{R}{F_1^2} = \alpha. \tag{4.3}$$

Equation (3.2) now becomes

$$\Psi_{\eta\eta\eta} + \lambda \frac{dG}{dS} \Psi_\eta^2 = \lambda G(S)(\Psi_\eta \Psi_{\eta S} - \Psi_S \Psi_{\eta\eta}) - \alpha G^3(S) \sin\beta = 0, \tag{4.4}$$

with the same boundary conditions, (3.3) and (3.4), as before.

The idea of this section is to use beds that are slowly varying with S in such a way that $\lambda G(S)(\Psi_\eta \Psi_{\eta S} - \Psi_S \Psi_{\eta\eta})$ becomes small, while $(dG/dS)\Psi_\eta^2$ remains $O(1)$. A naïve approach is to take $dG/dS = A(\epsilon S)$, where A is a general bounded function and ϵ is a small parameter. Then we might hope that Ψ would become a function of ϵS so that Ψ_S would be $O(\epsilon)$. Unfortunately G becomes $O(\epsilon^{-1})$ so the method fails. The film thickness must vary in such a way that $\partial/\partial S$ introduces a factor of $1/G$.

To achieve the object we introduce an auxiliary real variable z such that

$$\frac{dG}{dS} = H'(z), \tag{4.5}$$

and

$$\frac{dz}{dS} = \frac{\epsilon}{G}. \tag{4.6}$$

Then if Ψ is regarded as a function of z and η such that $\partial\Psi/\partial z$, $\partial\Psi/\partial\eta$ etc. are bounded, $\partial\Psi/\partial S = O(\epsilon/G)$ so that the nonlinear terms in the right-hand side of (4.4) are $O(\epsilon)$, regardless of how large G may become. Here $H(z)$ is a function at our disposal, which we choose in such a way that $S \to \pm\infty$ as $z \to \pm\infty$ and S is a monotonically increasing function of z. From (4.5) and (4.6) it is found that

$$\frac{dG}{dz} = \frac{G}{\epsilon} H'(z), \tag{4.7}$$

and hence a solution is

$$G = \exp\{\epsilon^{-1} H(z)\}. \tag{4.8}$$

Then from (4.6) it is found that

$$\epsilon S = \int_0^z \exp\{\epsilon^{-1} H(z_1)\}\, dz_1. \tag{4.9}$$

This is very similar to Fraenkel's theory for channels of slowly varying wall angle, in which a channel thickness proportional to $\exp(Q(\sigma)/\epsilon')$ appears.

Since we have non-dimensionalized by using the minimum value of the film thickness then G has a minimum value of unity. Therefore, from (4.8), we must choose $H(z) \geqslant 0$ for all z, so that G will be large when ϵ is small. Thus there will be rapid variations in G, regarded as a function of z. But since $\mathrm{d}G/\mathrm{d}S = H'(z)$ then a plot of G as a function of S will be entirely smooth for all values of S. The variable z is an 'extremely slow' variable. We must choose the smallest value of $H(z)$ to be zero, $H(z) \geqslant 0$, and in order to achieve semi-Poiseuille flow at $S = \pm\infty$ we choose $H'(\pm\infty) = 0$. Then from (4.9) it is seen that S is a monotonically increasing function of z, and as we vary z from $-\infty$ to $+\infty$, S will increase over the same range.

Assuming now that in (4.4) $\Psi = \phi(\eta, z) + O(\epsilon)$ we find

$$\phi_{\eta\eta\eta} + \lambda H'(z)\,\phi_\eta^2 + \alpha G^3 \sin\beta = 0, \tag{4.10}$$

where α is defined in (4.3). (In slender-channel flow the last term would be minus G^3 times the pressure gradient in the streamwise direction.) We choose the equation of the bed to be given by

$$\sin\beta = \frac{3f(z)}{\alpha G^3}, \tag{4.11}$$

where the factor $3/\alpha$ has been introduced for convenience, and then (4.10) becomes

$$\phi_{\eta\eta\eta} + \gamma(z)\,\phi_\eta^2 + 3f(z) = 0, \tag{4.12}$$

where

$$\gamma(z) = \lambda H'(z) = \lambda\frac{\mathrm{d}G}{\mathrm{d}S}. \tag{4.13}$$

The boundary conditions are

$$\phi = \phi_\eta = 0 \quad \text{on } \eta = 0, \tag{4.14}$$

$$\phi = 1, \quad \phi_{\eta\eta} = 0 \quad \text{on } \eta = 1. \tag{4.15}$$

Therefore z plays the role of a parameter in the ordinary differential equation (4.12) and if $\gamma(z)$ is specified we may solve the equation and boundary conditions to find $f(z)$ and $\phi(\eta, z)$, as in the problem (3.9)–(3.10). We may refer to figure 2 and table 1 for solutions since $f(z)$ is equivalent to μ^3 in (3.9).

A detailed consideration of the full Navier–Stokes equations and boundary conditions reveals that our approximation of this section, based on the boundary-layer equations, is indeed a first approximation to the full system provided that

$$\delta \ll \exp\left(\frac{-3H_{\mathrm{m}}}{\epsilon}\right), \tag{4.16}$$

where H_{m} is the maximum value of $H(z)$. To explain this we note that the second momentum equation written in terms of

$$u = U(S,Q) + O(\delta), \quad v = \delta V(S,Q) + O(\delta^2), \quad p = p_0 + \delta P_1(S,Q) + O(\delta^2),$$

yields

$$\frac{\partial P_1}{\partial Q} = \frac{-\alpha\cos\beta}{\lambda}. \tag{4.17}$$

Also, the full boundary conditions for continuity of stress at the free surface, when expanded in powers of δ with $R\delta = \lambda = O(1)$ show us that $P_1 = 0$ at $Q = G(S)$, where the free surface is $Q = G(S) + \delta G_1(S) + \dots$. Thus we find $P_1 = (\alpha\cos\beta/\lambda)\{G(S) - Q\}$ and hence $\partial P_1/\partial S$ is *not* zero. Therefore on writing the first momentum equation in terms of η and S in a similar way to (3.2), we find a term $\delta G^3\,\partial P_1/\partial S$. This must be negligible compared with unity, and for the special class of beds considered in this section this requirement leads to (4.16). It may be interpreted as stating that the film thickness is very small compared with the radius of curvature of the bed. Of course the Reynolds number R must be correspondingly large.

For the sake of brevity we have merely outlined the argument, but the full equations and boundary conditions have been written by the author in terms of η and z. For example, one of the terms neglected in the full equations in obtaining (3.2) was $\delta^2 \partial^2 U/\partial S^2$. In terms of $\Psi(S,\eta)$ we have

$$\frac{\partial^2 U}{\partial S^2} = \eta^2 \frac{(G')^2}{G^2} \Psi_{\eta\eta\eta} - 2\eta \frac{G'}{G^2} \Psi_{\eta\eta S} - h\frac{(G')^2}{G^3} \Psi_{\eta\eta} - \eta \frac{\mathrm{d}}{\mathrm{d}S}\left[\frac{G'}{G^2}\right] \Psi_{\eta\eta} + \frac{\Psi_{\eta SS}}{G} - \frac{\mathrm{d}}{\mathrm{d}S}\left[\frac{G'}{G^2}\right] \Psi_{\eta} - 2\frac{G'}{G^2}\Psi_{\eta S}, \quad (4.18)$$

where $G' = \mathrm{d}G/\mathrm{d}S$. Considering, for example the second term, writing $\Psi(S,\eta) = \phi(z,\eta)$ and using (4.5) and (4.6) we find

$$2\eta\left(\frac{G'}{G^2}\right)\Psi_{\eta\eta S} = \frac{2\eta\epsilon H'(z)\,\phi_{\eta\eta z}}{G^3}, \quad (4.19)$$

and similarly all the terms in (4.18) reduce to functions of η and z divided by G^3. Since, in obtaining (3.2) from (2.8) we multiplied by G^3 it is clear that here we neglected terms of $O(\delta^2)$ or smaller. Also the arclength parameter associated with S is $h_1 = 1+\delta Q\,\mathrm{d}\beta/\mathrm{d}S$ and this may be taken to be unity at first order.

We next explain how semi-Poiseuille flow is automatically achieved at $S = -\infty$, and give a geometrical interpretation of the constant α defined in (4.3). The equation of the bed is given by (4.11), where $f(z)$ is determined by (4.12), (4.14) and (4.15), provided $\gamma(z)$ is specified. We refer to figure 2 to explain the behaviour of $f(z)$. It should be noted that at positions where $\mathrm{d}G/\mathrm{d}S = 0$ we have $\gamma(z) = 0$ and hence $f(z) = 1$. Thus at the point of minimum film thickness we have $\gamma = 0$, $G = 1$ and $f = 1$. Let the angle of the bed at this point be β_1, then from (4.11) it is seen that

$$\alpha = \frac{3}{\sin\beta_1}. \quad (4.20)$$

We assume α is a constant independent of ϵ and δ.

Also, from (4.11) and figure 2, it is clear that

$$\sin\beta_0 = \frac{3}{\alpha G^3(-\infty)}, \quad (4.21)$$

where $\beta \to \beta_0$ as $S \to -\infty$. From the definitions (4.3) of α, and (4.2) of F_1 it follows that $F_1^2/F^2 = d^3/d_1^3$ and hence

$$F_1^2 = F^2 G^3(-\infty). \quad (4.22)$$

Using (4.21) and (4.22) in the definition $R/F_1^2 = \alpha$ it is found that

$$\frac{R}{F^2} = \frac{3}{\sin\beta_0}, \quad (4.23)$$

which is the required relation for semi-Poiseuille flow at $S = -\infty$. Of course, if the minimum film thickness occurs at $S = z = -\infty$ then $d = d_1$, $\beta_0 = \beta_1$, $F = F_1$ and $G(-\infty) = 1$ so that (4.20) is the same as (4.21).

We now show how the theory is applied in several examples.

Example 1

Here we specify β_0, the angle of the bed at $S = -\infty$, and choose $H(-\infty) = 0$, $H(z)$ monotonically increasing with z and $H(\infty) = H_1 > 0$. Thus the minimum film thickness is at $S = z = -\infty$, and $G(-\infty) = 1$. Hence from (4.20) we have $\alpha = 3/\sin\beta_0$ and the equation of the bed is

$$\sin\beta = \frac{f(z)\sin\beta_0}{G^3}, \tag{4.24}$$

where G is given by (4.8), with S given by (4.9). We suppose $\lambda H'(z) < 1.81$ for all z. Then the values of $\gamma(z)$ in (4.12) are positive but less than 1.81, and from figure 2 it is seen that $f(z)$ decreases from unity at $S = -\infty$ to reach a minimum value, which is positive, and then increases again to unity at $S = \infty$. In figure 4 we show schematically how $\lambda H'(z)$ and $f(z)$ vary with z (curves B).

The streamwise velocity profiles are of class B (see §3) and change from semi-Poiseuille flow at $S = -\infty$ (where $f(z) = 1$, $G = 1$ and $\beta = \beta_0$), through a range of Jeffery–Hamel profiles for $0 < \gamma < 1.81$, then back again to semi-Poiseuille flow at $S = \infty$. From (4.24) we see that $\sin\beta = \sin\beta_0 \exp(-3H_1/\epsilon)$ at $S = \infty$. From (4.24), (4.5) and (4.6)

$$(\cos\beta)\frac{d\beta}{dS} = \left\{\frac{-3f(z)H'(z) + \epsilon f'(z)}{G^4}\right\}\sin\beta_0, \tag{4.25}$$

and since $H'(z)$ and $f(z)$ are positive the curvature is negative for sufficiently small values of ϵ, i.e. the bed is concave upwards and β remains positive throughout. The flow may be described as 'divergent flow on a concave downward-sloping bed' and we note that in this case, although the velocity profiles may become considerably 'sharper' than semi-Poiseuille flow, no points of inflexion occur and no regions of reversed flow exist.

Example 2

Again, as in example 1 we choose $H(-\infty) = 0$, $H(\infty) = H_1 > 0$ and H monotonically increasing with z, with the equation of the bed being given by (4.22). However, we now assume that $H'(z)$ reaches a maximum value at $z = 0$, say, with

$$1.81 < \lambda H'(0) < 5.46. \tag{4.26}$$

By reference to figure 2 and figure 4 (curves C) we see that $f(z)$ now takes negative values in a range of z and hence, from (4.11), $\sin\beta$ becomes negative in this range of z. Thus the bed takes the form illustrated in figure 5. It is a generally downward-sloping concave bed, with a hump on which β becomes negative. The angle of the bed is β_0 at $S = -\infty$; and at $S = \infty$, $\sin\beta = (\sin\beta_0)\exp(-3H_1/\epsilon)$. If we take λ large enough then $\lambda H'(z) = \gamma(z)$ is greater than 4.71 in a range of z, so that profiles of class C with $\gamma > 4.71$ (see §3) occur. Thus a region of reversed flow exists near the wall. Separation and reattachment occur smoothly. There are no singularities in the flow field.

In table 2 we give approximate values of S, X, Y and G for the case when $H(z) = \frac{1}{4}(1 + \tanh z)$, with $\lambda = 20$, $\epsilon = 0.4$ and $\sin\beta_0 = 0.5$. Here X, Y are the Cartesian coordinates, with X measured horizontally and Y vertically upwards, the origin being at $S = 0$.

The bed and film thickness are sketched in figure 5, with the actual film thickness

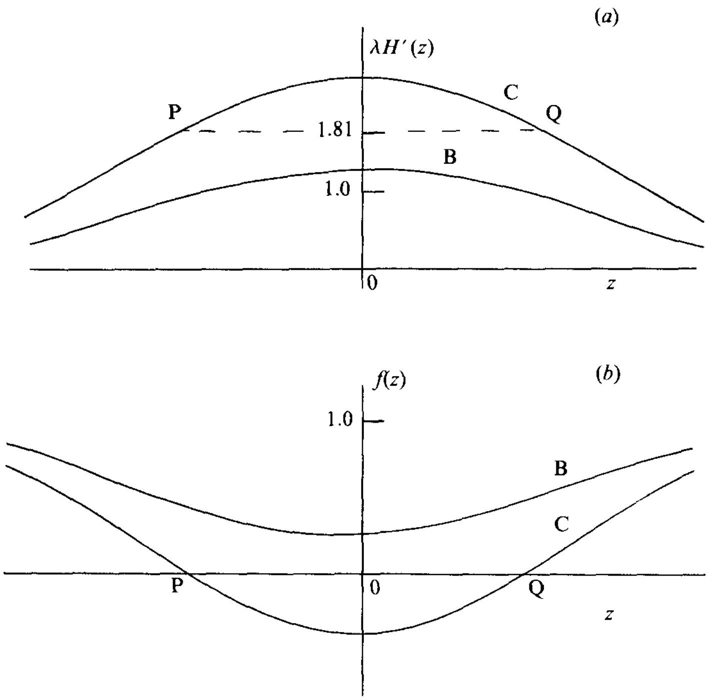

FIGURE 4. Behaviour of (a) $\lambda H'(z)$ and (b) $f(z)$ in example 1 (curves B) and example 2 (curves C).

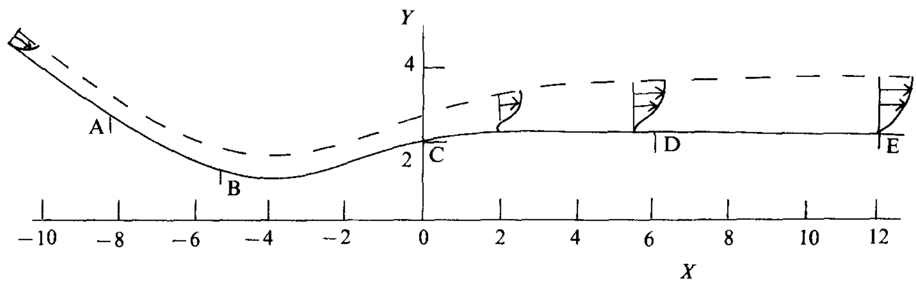

FIGURE 5. Sketch of flow for $H(z) = 0.25\,(1+\tanh z)$, $\lambda = 20$, $\epsilon = 0.4$. At point A, $\gamma = 0.05$; at B, $\gamma = 2.1$; at C, $\gamma = 5.0$; at D, $\gamma = 2.1$; at E, $\gamma = 0.51$. The true film thickness is 2.5δ times that sketched. The velocity profiles are very approximate.

being 2.5δ times that shown on the diagram, since to show the actual thickness for say $\delta = 0.1$ would make it too small to represent diagrammatically together with the shape of the bed.

Example 3

We choose $H(-\infty) = H_2 > 0$, $H(\infty) = 0$ and H monotonically decreasing with z. Thus, from (4.8), G has its minimum value at $S = \infty$ and the angle there is β_1. The equation of the bed is

$$\sin\beta = \frac{f(z)\sin\beta_1}{G^3}.$$

Since $H'(z)$ is generally negative, the values of $\gamma(z)$ are negative, and from figure

z	S	G	X	Y	β	γ
$-\infty$	$-\infty$	1.00	$-\infty$	∞	0.523	0
-3.0	-8.80	1.00	-8.34	2.797	0.504	0.049
-2.0	-6.27	1.02	-6.09	1.668	0.393	0.353
-1.0	-3.60	1.16	-3.49	1.147	-0.058	2.10
0	0	1.87	0	2.0	-0.282	5.0
1.0	6.18	3.01	6.16	2.389	-0.003	2.10
2.0	14.3	3.41	14.3	2.337	$+0.010$	0.353
3.0	22.9	3.49	22.9	2.241	0.011	0.049
∞	∞	3.49	∞	$-\infty$	0.012	0

TABLE 2. Some values of parameters and coordinates when $H(z) = 0.25\ (1+\tanh z)$, $\lambda = 20$, $\epsilon = 0.4$, $\sin\beta_0 = 0.5$ for example 3 of §4

2 the values of $f(z)$ are positive. The Jeffery–Hamel flows attained are of class A (see §3), and are 'flatter' than semi-Poiseuille flow. This is a downward-sloping, convex bed on which the film thickness decreases as S increases.

Other examples may easily be constructed. For example, it is possible to choose $H(z)$ in such a way that the flow tends to a Jeffery–Hamel flow as $S \to \infty$. Or it is possible to choose $H(z)$ to be zero at some finite value of z, say $z = 0$, giving a minimum film thickness at $S = 0$.

In the next section we discuss further points about the approximation, and whether or not these flows are likely to be attained experimentally.

5. Further discussion

5.1. *Singularities*

The type of bed discussed in §4 is rather special. The question may be asked 'Is the theory relevant to more usual beds whose equations are $\sin\beta = h(S)$?', so that the governing equation and boundary conditions are (2.8)–(2.12). The evidence from calculations on channel flows is encouraging. Eagles & Smith (1980) calculated steady-state flows in channels whose walls were given by $y = \pm H(X)$, where $X = \epsilon x$ and $R\epsilon = \lambda = O(1)$ as $\epsilon \to 0$. This leads to a problem identical with the present problem, except that a scaled pressure gradient replaces the gravitational term. In fact Eagles & Smith use a form of the boundary-layer equation

$$F_{\eta\eta\eta} + \lambda H(X)\,(F_{\eta\eta}F_X - F_\eta F_{\eta X}) + \lambda H'(X)\,F_\eta^2 - \lambda H^3(X)\frac{dP}{dX} = 0$$

where $H(X) = 1 + \frac{1}{2}\tanh X$ is specified as the channel width. The boundary conditions on the stream function obtained are the same as in the present paper if we take the flow field over half the channel width. The pressure gradient dP/dX is determined as part of the solution and for values of λ greater than about 6 it is found that dP/dX becomes positive over a limited range of X. Thus, writing $dP/dX = T(X)$, if in (3.2) we set $\sin\beta = -\frac{1}{3}\lambda T(S)\sin\beta_0$ it reduces to the same form as the equation used by Eagles & Smith. Therefore the solutions of Eagles & Smith are applicable to the present problem of thin-film boundary-layer flow, and wherever $T(S)$ becomes positive $\sin\beta$ becomes negative. There are no reported singularities at separation or elsewhere for calculations up to $\lambda = 18$. The flows found include some cases with reversed flow near the walls, qualitatively similar to the Jeffery–Hamel solutions with $\gamma > 4.71$.

Merkin (1973), in his calculation of boundary-layer flows over curved beds, postulated a singularity at separation, which seemed to be indicated by some of his numerical results, inasmuch as the streamwise gradient of the skin friction was becoming large as the separation point was approached. However, the work of Eagles & Smith mentioned above shows that the flow does not necessarily have a singularity at separation. Indeed it would appear unlikely that a singularity will in general appear at separation, since there is nothing especially remarkable about the case of $H = 1+\frac{1}{2}\tanh X$. While it is true that the slowly varying assumption made about the flow in §4 may, by its nature, exclude certain singularities, it is of interest to note that this theory predicts a singularity at $\gamma = \lambda\, \mathrm{d}G/\mathrm{d}S = 5.46$, rather than at the separation value of $\gamma = 4.71$. The conclusion we make is that for the ordinary boundary-layer equations there is not necessarily a singularity at separation in flow over a curved bed, and for the slowly varying theory of §4 there is definitely not a singularity at separation. Wang (1984) also gives an example of flow with separation and reattachment of the main stream over a curved bed, though this is with non-zero surface tension and is on a downward-sloping curved bed.

5.2. *Stability*

The question of the stability of the flows of §§3 and 4 is interesting. According to long-wave instability theory (see Yih 1963), for straight beds the critical Reynold number is $R_{c0} = \frac{5}{6}\cot\beta$, where β is the angle of inclination of the bed. Now if β approaches zero, R_{c0} approaches infinity. On a uniformly straight bed it is unrealistic to take $\beta = 0$, but on curved beds the local β may well become negative. In such cases the velocity profiles, however, are quite different from semi-Poiseuille flow and may well be more unstable. There may be a destabilizing effect due to the change in the streamwise velocity profile for the divergent flows of §§3 and 4, but a stabilizing effect due to the small, or negative values of β. It seems possible that these two effects may interact to make the flows described in §§3 and 4 stable for some finite range of R, even in the cases where β becomes negative. This however, is speculation, and detailed calculations are needed. This work is under way and results will be reported at a later date. The author has been able to find no references to theoretical work on stability of flow on a curved bed.

5.3. *Conclusions*

(i) We have demonstrated the existence of some exact solutions of the boundary-layer equations over certain curved beds with $\sin\beta = K/S^3$ for a limited range of S. This work is described in §3.

(ii) These exact solutions, which are of Jeffery–Hamel type, are shown in §4 to occur as a first approximation at every station for certain special beds of infinite streamwise extent.

(iii) For the flows of §4, separation and reattachment of the main stream can occur with no singularities.

(iv) We believe the specialized results of §4 give a guide to the types of flow that may occur over general curved beds with equations $\beta = h(S)$.

(v) The investigation of the stability of these flows would be of interest.

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